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# Triple positive pseudo-symmetric solutions to a four-point boundary value problem with $p$-Laplacian[☆]

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**Abstract**

This work deals with the existence of triple positive pseudo-symmetric solutions for the one-dimensional $p$-Laplacian

$$(\phi_p(u'))'(t) + q(t)f(t, u(t), u'(t)) = 0, \quad t \in (0, 1),$$
$$u(0) - \beta u'(\xi) = 0, \qquad u(\xi) - \delta u'(\eta) = u(1) + \delta u'(1 + \xi - \eta),$$

where $\phi_p(s) = |s|^{p-2} \cdot s$, $p > 1$. By means of a fixed point theorem due to Avery and Peterson, sufficient conditions are obtained that guarantee the existence of at least three positive pseudo-symmetric solutions to the above boundary value problem. The interesting point is that the nonlinear term is involved with the first-order derivative explicitly.


*Keywords:* Positive pseudo-symmetric solutions; $p$-Laplacian; Cone; Four-point boundary value problem; Fixed point theorem

## 1. Introduction

In this work, we consider the following four-point boundary value problem (BVP for short) with a $p$-Laplacian operator:

$$(\phi_p(u'(t)))' + q(t)f(t, u(t), u'(t)) = 0, \quad t \in (0, 1), \tag{1}$$

$$u(0) - \beta u'(\xi) = 0, \qquad u(\xi) - \delta u'(\eta) = u(1) + \delta u'(1 + \xi - \eta), \tag{2}$$

where $\phi_p(s) = |s|^{p-2} \cdot s$, $p > 1$, $(\phi_p)^{-1} = \phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$, $\beta > 0$, $\delta \geq 0$, $\xi, \eta \in (0, 1)$ is prescribed and $\xi < \eta$.

Equations of the above form occur in the study of the $n$-dimensional equation, non-Newtonian fluid theory and the turbulent flow of a gas in a porous medium [8]. For when the nonlinear term $f$ does not depend on the first-order derivative, Eq. (1) has been studied extensively, and the existence and multiplicity results are available in the

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literature [1–5]. Recently, in [6], Li and Shen considered the multiplicity for the BVP

$$(g(u'))' + a(t)f(u) = 0, \quad 0 < t < 1, \tag{3}$$

$$u(0) - B_0(u'(\eta)) = 0, \qquad u'(1) = 0, \tag{4}$$

or

$$u'(0) = 0, \qquad u(1) + B_1(u'(\eta)) = 0, \tag{5}$$

and by applying the five-functionals fixed point theorem, the authors proved that BVP (3) and (4) or (5) has at least three positive solutions.

However, there are few papers dealing with the existence of triple positive solutions for when the nonlinear term $f$ is involved in the first-order derivative explicitly. In [7], Bai considered the multiplicity for Eq. (1) together with one of the following boundary conditions:

$$\alpha\phi_p(x(0)) - \beta\phi_p(x'(0)) = 0, \qquad \gamma\phi_p(x(1)) + \delta\phi_p(x'(1)) = 0, \tag{6}$$

or

$$x(0) - g_1(x'(0)) = 0, \qquad x(1) + g_2(x'(1)) = 0, \tag{7}$$

by means of the Avery–Peterson fixed point theorem.

On the other hand, the existence of positive pseudo-symmetric solutions was studied by some authors. Avery and Henderson [3] consider the existence of three positive pseudo-symmetric solutions for the following problem:

$$(\phi_p(u'))'(t) + q(t)f(t, u(t)) = 0, \quad t \in (0, 1), \tag{8}$$

$$u(0) = 0, \qquad u(\eta) = u(1). \tag{9}$$

The definition of pseudo-symmetric was introduced in that paper.

As we know, when the nonlinear term $f$ is involved in the first-order derivative, difficulties arise immediately. In this work, we use a fixed point theorem due to Avery and Peterson to overcome the difficulties. To the best of the knowledge of the authors, no work has been done for BVP (1) and (2) by use of the Avery and Peterson fixed point theorem. The aim of this work is to fill the gap in the relevant literature.

Throughout, it is assumed that:

($H_1$) $f \in C([0, 1] \times [0, \infty) \times R, [0, \infty))$, $f(t, 0, 0) \neq 0$ identically on $[0, 1]$ and $f(t, x, y) = f(1 + \xi - t, x, -y)$, $(t, x, y) \in [\xi, 1] \times [0, \infty) \times R$;

($H_2$) $q(t) \in L^1(0, 1)$ nonnegative and $q(t) = q(1 + \xi - t)$, a.e. $t \in [\xi, 1]$, and $q(t) \not\equiv 0$ on any subinterval of $[0, 1]$.

## 2. Preliminary

For the convenience of the reader, we provide some background material from the theory of cones in Banach spaces.

**Definition 2.1.** For $\xi \in (0, 1)$, a function $u \in E$ is said to be pseudo-symmetric if $u$ is symmetric over the interval $[\xi, 1]$. That is, for $t \in [\xi, 1]$ we have $u(t) = u(1 + \xi - t)$.

**Remark 2.1.** For $\xi \in (0, 1)$, if $u \in E$ is pseudo-symmetric, we have $u'(t) = -u'(1 + \xi - t)$, $t \in [\xi, 1]$.

**Definition 2.2.** The map $\alpha$ is said to be a nonnegative continuous concave functional on a cone $K$ of a real Banach space $E$ provided that $\alpha : K \to [0, \infty)$ is continuous and

$$\alpha(tx + (1 - t)y) \geq t\alpha(x) + (1 - t)\alpha(y)$$

for all $x, y \in K$ and $0 \leq t \leq 1$. Similarly, we say the map $\gamma$ is a nonnegative continuous convex functional on a cone $K$ of a real Banach space $E$ provided that $\gamma : K \to [0, \infty)$ is continuous and

$$\gamma(tx + (1 - t)y) \leq t\gamma(x) + (1 - t)\gamma(y)$$

for all $x, y \in K$ and $0 \leq t \leq 1$.

Let $\gamma$ and $\theta$ be nonnegative continuous convex functionals on $K$, $\alpha$ be a nonnegative continuous concave functional on $K$, and $\psi$ be a nonnegative continuous functional on $K$. Then for positive real numbers $a, b, c$, and $d$, we define the following convex sets:

$$K(\gamma, d) = \{x \in K \mid \gamma(x) < d\},$$
$$K(\gamma, \alpha, b, d) = \{x \in K \mid b \leq \alpha(x), \gamma(x) \leq d\},$$
$$K(\gamma, \theta, \alpha, b, c, d) = \{x \in K \mid b \leq \alpha(x), \theta(x) \leq c, \gamma(x) \leq d\},$$
$$R(\gamma, \psi, a, d) = \{x \in K \mid a \leq \psi(x), \gamma(x) \leq d\}.$$

The following fixed point theorem due to Avery and Peterson is fundamental in the proofs of our main results.

**Theorem 2.1** ([4]). *Let $K$ be a cone in a Banach space $E$. Let $\gamma$ and $\theta$ be nonnegative continuous convex functionals on $K$, $\alpha$ be a nonnegative continuous concave functional on $K$, and $\psi$ be a nonnegative continuous functional on $K$ satisfying $\psi(\lambda x) \leq \lambda\psi(x)$ for $0 \leq \lambda \leq 1$, such that for some positive numbers $M$ and $d$,*

$$\alpha(x) \leq \psi(x) \quad \text{and} \quad \|x\| \leq M\gamma(x) \tag{10}$$

*for all $x \in \overline{K(\gamma, d)}$. Suppose*

$$T : \overline{K(\gamma, d)} \to \overline{K(\gamma, d)}$$

*is completely continuous and there exist positive numbers $a, b$, and $c$ with $a < b$ such that*

($S_1$) $\{x \in K(\gamma, \theta, \alpha, b, c, d) | \alpha(x) > b\} \neq \emptyset$ *and* $\alpha(Tx) > b$ *for* $x \in K(\gamma, \theta, \alpha, b, c, d)$;
($S_2$) $\alpha(Tx) > b$ *for* $x \in K(\gamma, \alpha, b, d)$ *with* $\theta(Tx) > c$;
($S_3$) $0 \notin R(\gamma, \psi, a, d)$ *and* $\psi(Tx) < a$ *for* $x \in R(\gamma, \psi, a, d)$ *with* $\psi(x) = a$.

*Then $T$ has at least three fixed points $x_1, x_2, x_3 \in \overline{K(\gamma, d)}$ such that*

$$\gamma(x_i) \leq d \quad \text{for } i = 1, 2, 3,$$
$$b < \alpha(x_1),$$
$$a < \psi(x_2), \quad \text{with } \alpha(x_2) < b,$$

*and*

$$\psi(x_3) < a.$$

## 3. Existence result

In this section, we impose growth condition on $f$ which allow us to apply Theorem 2.1 to the boundary value problem (1) and (2). It follows from ($H_2$) that there exists a constant $\omega \in (0, \frac{1}{2})$ such that $0 < \int_{\omega}^{1-\omega} q(t)\mathrm{d}t < +\infty$.

Let $X = C^1[0, 1]$ be endowed with the maximum norm,

$$\|u\| = \max\left\{\max_{0\leq t\leq 1} |u(t)|, \max_{0\leq t\leq 1} |u'(t)|\right\}.$$

From the fact that $(\phi_p(u'(t)))' = -q(t)f(t, u(t), u'(t)) \leq 0$, we know that $u$ is concave on $[0, 1]$. So, define the cone $K$ by

$$K = \{u \in X | u(t) \geq 0, u(0) - \beta u'(\xi) = 0, u \text{ is concave on } [0, 1] \text{ and symmetric on} [\xi, 1]\}.$$

Let the nonnegative continuous concave functional $\alpha$, the nonnegative continuous convex functional $\theta, \gamma$ and the nonnegative continuous functional $\psi$ be defined on the cone $K$ by

$$\gamma(u) = \max_{0\leq t\leq 1} |u'(t)|, \qquad \psi(u) = \theta(u) = \max_{0\leq t\leq 1} |u(t)|, \qquad \alpha(u) = \min_{\omega\leq t\leq 1-\omega} |u(t)|, \tag{11}$$

for $u \in K$.

**Lemma 3.1** ([9]). *Let $u \in K$, $\omega \in (0, \frac{1}{2})$; then $u(t) \geq \omega \max_{0\leq t\leq 1} |u(t)|$, $t \in [\omega, 1 - \omega]$.*

**Lemma 3.2.** *For $u \in K$,*

$$\max_{0 \le t \le 1} |u(t)| \le (1+\beta) \max_{0 \le t \le 1} |u'(t)|. \tag{12}$$

**Proof.** By the concavity of $u$, there is

$$u(t) - u(0) \le u'(0) \le \max_{0 \le t \le 1} |u'(t)|, \quad t \in [0,1]. \tag{13}$$

On the other hand, taking into account that $u$ is nonnegative, we have

$$u(0) = \beta(u'(\xi)) \le \beta \max_{0 \le t \le 1} |u'(t)|.$$

Thus,

$$\max_{0 \le t \le 1} |u(t)| \le (1+\beta) \max_{0 \le t \le 1} |u'(t)|. \quad \square$$

With Lemmas 3.1 and 3.2, for all $u \in K$, for the functionals defined above there hold the relations

$$\omega\theta(u) \le \alpha(u) \le \theta(u) = \psi(u), \qquad \|u\| = \max\{\theta(u), \gamma(u)\} \le (1+\beta)\gamma(u). \tag{14}$$

Therefore, the condition (10) of Theorem 2.1 is satisfied.

Define an operator $T : K \to X$ by

$$(Tu)(t) = \begin{cases} \beta\phi_q\left(\int_\xi^{\frac{1+\xi}{2}} q(\tau)f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right) + \int_0^t \phi_q\left(\int_s^{\frac{1+\xi}{2}} q(\tau)f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\mathrm{d}s, \\ \quad 0 \le t \le \dfrac{1+\xi}{2}, \\ \beta\phi_q\left(\int_\xi^{\frac{1+\xi}{2}} q(\tau)f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right) + \int_0^\xi \phi_q\left(\int_s^{\frac{1+\xi}{2}} q(\tau)f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\ \quad + \int_t^1 \phi_q\left(\int_{\frac{1+\xi}{2}}^s q(\tau)f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\mathrm{d}s, \quad \dfrac{1+\xi}{2} \le t \le 1. \end{cases} \tag{15}$$

Obviously, $Tu \in X$, and we can prove that each fixed point of $T$ is a solution of problem (1) and (2).

**Lemma 3.3.** *$T : K \to K$ is completely continuous.*

**Proof.** For any $t \in [\xi, \frac{1+\xi}{2}]$, we have $1+\xi-t \in [\frac{1+\xi}{2}, 1]$. Therefore,

$$\begin{aligned} (Tu)(1+\xi-t) &= \beta\phi_q\left(\int_\xi^{\frac{1+\xi}{2}} q(\tau)f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right) + \int_0^\xi \phi_q\left(\int_s^{\frac{1+\xi}{2}} q(\tau)f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\ &\quad + \int_{1+\xi-t}^1 \phi_q\left(\int_{\frac{1+\xi}{2}}^s q(\tau)f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\ &= \beta\phi_q\left(\int_\xi^{\frac{1+\xi}{2}} q(\tau)f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right) + \int_0^\xi \phi_q\left(\int_s^{\frac{1+\xi}{2}} q(\tau)f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\ &\quad + \int_\xi^t \phi_q\left(\int_s^{\frac{1+\xi}{2}} q(\tau)f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\ &= \beta\phi_q\left(\int_\xi^{\frac{1+\xi}{2}} q(\tau)f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right) + \int_0^t \phi_q\left(\int_s^{\frac{1+\xi}{2}} q(\tau)f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\ &= (Tu)(t). \end{aligned} \tag{16}$$

So, $Tu$ is symmetric on $[\xi, 1]$ with respect to $\frac{1+\xi}{2}$. Obviously, $(Tu)(t) \ge 0$, $Tu$ is concave on $[0,1]$, $(Tu)(0) - \beta(Tu)'(\xi) = 0$. Therefore, $T : K \to K$. Furthermore, standard arguments yield that $T$ is completely continuous. $\square$

Let

$$C=\phi_q\left(\int_0^{\frac{1+\xi}{2}} q(\tau)\mathrm{d}\tau\right),\qquad L=\beta\phi_q\left(\int_\xi^{\frac{1+\xi}{2}} q(\tau)\mathrm{d}\tau\right)+\int_0^{\frac{1+\xi}{2}}\phi_q\left(\int_s^{\frac{1+\xi}{2}} q(\tau)\mathrm{d}\tau\right)\mathrm{d}s,$$

$$N=\int_\omega^{\frac{1}{2}}\phi_q\left(\int_s^{\frac{1}{2}} q(\tau)\mathrm{d}\tau\right)\mathrm{d}s.$$

**Theorem 3.4.** *Assume that* $(\mathrm{H}_1)$, $(\mathrm{H}_2)$ *hold. Let* $0<a<b\le\omega d(\frac{1+\xi-\omega}{1+\xi})$, *and suppose that* $f$ *satisfies the following conditions:*

$(\mathrm{A}_1)$ $f(t,h,k)\le\phi_p\left(\frac{d}{C}\right)$, *for* $(t,h,k)\in[0,1]\times[0,(1+\beta)d]\times[-d,d]$;

$(\mathrm{A}_2)$ $f(t,h,k)>\phi_p\left(\frac{b}{\omega N}\right)$, *for* $(t,h,k)\in[\omega,1-\omega]\times[b,\frac{b(1+\xi)^2+4b\beta(1-\xi)}{4\omega(1+\xi-\omega)}+\frac{b}{\omega}]\times[-d,d]$;

$(\mathrm{A}_3)$ $f(t,h,k)<\phi_p\left(\frac{a}{L}\right)$, *for* $(t,h,k)\in[0,1]\times[0,a]\times[-d,d]$.

*Then the boundary value problem* (1) *and* (2) *has at least three positive pseudo-symmetric solutions* $u_1$, $u_2$ *and* $u_3$ *such that*

$$\max_{0\le t\le1}|u_i'(t)|\le d\quad \text{for } i=1,2,3,$$

$$b<\min_{\omega\le t\le1-\omega}|u_1(t)|,\qquad \max_{0\le t\le1}|u_1(t)|\le(1+\beta)d,$$

$$a<\max_{0\le t\le1}|u_2(t)|<\frac{b(1+\xi)^2+4b\beta(1-\xi)}{4\omega(1+\xi-\omega)}+\frac{b}{\omega},\quad \text{with}\ \min_{\omega\le t\le1-\omega}|u_2(t)|<b,$$

*and*

$$\max_{0\le t\le1}|u_3(t)|<a.$$

**Proof.** Problem (1) and (2) has a solution $u=u(t)$ if and only if $u$ solves the operator equation $u=Tu$. Thus we set out to prove that $T$ satisfies the Avery–Peterson fixed point theorem which will prove the existence of three fixed points of $T$ which satisfy the conclusion of the theorem.

For $u\in\overline{K(\gamma,d)}$, there is $\gamma(u)=\max_{0\le t\le1}|u'(t)|\le d$. With Lemma 3.2, there is $\max_{0\le t\le1}|u(t)|\le(1+\beta)d$; then condition $(\mathrm{A}_1)$ implies $f(t,u(t),u'(t))\le\phi_p\left(\frac{d}{C}\right)$. On the other hand, for $u\in K$, there is $Tu\in K$; then $Tu$ is concave on [0,1], and $\max_{0\le t\le1}|(Tu)'(t)|=(Tu)'(0)$, so

$$\gamma(Tu)=\max_{0\le t\le1}|(Tu)'(t)|=\phi_q\left(\int_0^{\frac{1+\xi}{2}} q(\tau)f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\le\frac{d}{C}\phi_q\left(\int_0^{\frac{1+\xi}{2}} q(\tau)\mathrm{d}\tau\right)=\frac{d}{C}C=d.$$

Therefore, $T:\overline{K(\gamma,d)}\to\overline{K(\gamma,d)}$.

To check condition $(\mathrm{S}_1)$ of Theorem 2.1, we choose $u(t)=\frac{bt(1+\xi-t)}{\omega(1+\xi-\omega)}+\frac{b\beta(1-\xi)}{\omega(1+\xi-\omega)}$, $0\le t\le1$. It is easy to see that $u(t)=\frac{bt(1+\xi-t)}{\omega(1+\xi-\omega)}+\frac{b\beta(1-\xi)}{\omega(1+\xi-\omega)}\in K(\gamma,\theta,\alpha,b,\frac{b(1+\xi)^2+4b\beta(1-\xi)}{4\omega(1+\xi-\omega)}+\frac{b}{\omega},d)$ and $\alpha(u)=\alpha(\frac{bt(1+\xi-t)}{\omega(1+\xi-\omega)}+\frac{b\beta(1-\xi)}{\omega(1+\xi-\omega)})>b$, and so $\{u\in K(\gamma,\theta,\alpha,b,\frac{b(1+\xi)^2+4b\beta(1-\xi)}{4\omega(1+\xi-\omega)}+\frac{b}{\omega},d)|\alpha(u)>b\}\ne\emptyset$. Hence, for $u\in K(\gamma,\theta,\alpha,b,\frac{b(1+\xi)^2+4b\beta(1-\xi)}{4\omega(1+\xi-\omega)}+\frac{b}{\omega},d)$, there is $b\le u(t)\le\frac{b(1+\xi)^2+4b\beta(1-\xi)}{4\omega(1+\xi-\omega)}+\frac{b}{\omega}$, $|u'(t)|\le d$ for $\omega\le t\le1-\omega$.

Thus, by condition $(\mathrm{A}_2)$ of this theorem, we have $f(t,u(t),u'(t))>\phi_p\left(\frac{b}{\omega N}\right)$ for $\omega\le t\le1-\omega$, and combining the conditions of $\alpha$ and $K$, we have

$$\begin{aligned}\alpha(Tu)&=\min_{\omega\le t\le\omega}|(Tu)(t)|\ge\omega\max_{0\le t\le1}|Tu(t)|=\omega\left(\beta\phi_q\left(\int_\xi^{\frac{1+\xi}{2}} q(\tau)f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\right.\\&\quad\left.+\int_0^{\frac{1+\xi}{2}}\phi_q\left(\int_s^{\frac{1+\xi}{2}} q(\tau)f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\mathrm{d}s\right)\\&\ge\omega\left(\int_0^{\frac{1+\xi}{2}}\phi_q\left(\int_s^{\frac{1+\xi}{2}} q(\tau)f(\tau,u(\tau),u'(\tau))\mathrm{d}\tau\right)\mathrm{d}s\right)\end{aligned}$$

$$
\begin{aligned}
&\geq \omega\left(\int_{\omega}^{\frac{1}{2}} \phi_q\left(\int_s^{\frac{1}{2}} q(\tau) f(\tau, u(\tau), u'(\tau))\mathrm{d}\tau\right)\mathrm{d}s\right) \\
&> \omega\frac{b}{\omega N}\int_{\omega}^{\frac{1}{2}} \phi_q\left(\int_s^{\frac{1}{2}} q(\tau)\mathrm{d}\tau\right)\mathrm{d}s = \frac{b}{N}N = b
\end{aligned}
$$

i.e., $\alpha(Tu) > b$ for all $u \in K(\gamma, \theta, \alpha, b, \frac{b(1+\xi)^2+4b\beta(1-\xi)}{4\omega(1+\xi-\omega)} + \frac{b}{\omega}, d\}$. This shows that condition ($S_1$) of Theorem 2.1 is satisfied.

Secondly, with (14), we have

$$
\alpha(Tu) \geq \omega\theta(Tu) > \omega\left(\frac{b(1+\xi)^2+4b\beta(1-\xi)}{4\omega(1+\xi-\omega)} + \frac{b}{\omega}\right) = b
$$

$$
\text{for all } u \in K(\gamma, \alpha, b, d) \text{ with } \theta(Tu) > \frac{b(1+\xi)^2+4b\beta(1-\xi)}{4\omega(1+\xi-\omega)} + \frac{b}{\omega}.
$$

Thus, condition ($S_2$) of Theorem 2.1 is satisfied.

Finally, we show that condition ($S_3$) of Theorem 2.1 also holds. Clearly, as $\psi(0) = 0 < a$, there holds $0 \notin R(\gamma, \psi, a, d)$. Suppose that $u \in R(\gamma, \psi, a, d)$ with $\psi(u) = a$.

Then, by the condition ($A_3$) of this theorem,

$$
\begin{aligned}
\psi(Tu) = \max_{0\leq t\leq 1} |(Tu)(t)| &= \beta\phi_q\left(\int_{\xi}^{\frac{1+\xi}{2}} q(\tau) f(\tau, u(\tau), u'(\tau))\mathrm{d}\tau\right) \\
&\quad + \int_0^{\frac{1+\xi}{2}} \phi_q\left(\int_s^{\frac{1+\xi}{2}} q(\tau) f(\tau, u(\tau), u'(\tau))\mathrm{d}\tau\right)\mathrm{d}s \\
&< \frac{a}{L}\left(\beta\phi_q\left(\int_{\xi}^{\frac{1+\xi}{2}} q(\tau)\mathrm{d}\tau\right) + \int_0^{\frac{1+\xi}{2}} \phi_q\left(\int_s^{\frac{1+\xi}{2}} q(\tau)\mathrm{d}\tau\right)\mathrm{d}s\right) \\
&= \frac{a}{L}L = a.
\end{aligned}
$$

So, the condition ($S_3$) of Theorem 2.1 is satisfied. Therefore, an application of Theorem 2.1 implies that the boundary value problem (1) and (2) has at least three positive pseudo-symmetric solutions $u_1$, $u_2$ and $u_3$ such that

$$
\begin{aligned}
&\max_{0\leq t\leq 1} |u_i'(t)| \leq d \quad \text{for } i = 1, 2, 3, \\
&b < \min_{\omega\leq t\leq 1-\omega} |u_1(t)|, \qquad \max_{0\leq t\leq 1} |u_1(t)| \leq (1+\beta)d, \\
&a < \max_{0\leq t\leq 1} |u_2(t)| < \frac{b(1+\xi)^2+4b\beta(1-\xi)}{4\omega(1+\xi-\omega)} + \frac{b}{\omega}, \quad \text{with } \min_{\omega\leq t\leq 1-\omega} |u_2(t)| < b,
\end{aligned}
$$

and

$$
\max_{0\leq t\leq 1} |u_3(t)| < a.
$$

The proof is complete. □

**Example 3.1.** Consider the boundary value problem

$$
(|u'|u')' + f(t, u(t), u'(t)) = 0, \quad 0 < t < 1, \tag{17}
$$

$$
u(0) - \frac{1}{4}u'\left(\frac{1}{3}\right) = 0, \qquad u\left(\frac{1}{3}\right) - 8u'\left(\frac{1}{2}\right) = u(1) + 8u'\left(\frac{5}{6}\right), \tag{18}
$$

where

$$f(t,h,k)=\begin{cases}\dfrac{1}{100}t\left(\dfrac{4}{3}-t\right)+69h^8+\dfrac{1}{100}\left(\dfrac{k}{60\,000}\right)^2, & \text{for } 0\le h\le 10,\\ \dfrac{1}{100}t\left(\dfrac{4}{3}-t\right)+69\cdot 6^8+\dfrac{1}{100}\left(\dfrac{k}{60\,000}\right)^2, & \text{for } h>10.\end{cases}$$

We notice that $p = 3, q = \frac{3}{2}, \xi = \frac{1}{3}, \eta = \frac{1}{2}, \beta = \frac{1}{4}, \delta = 8$; it follows from a direct calculation that $C = \frac{\sqrt{6}}{3},\ L = \frac{\sqrt{3}}{12} + \frac{4\sqrt{6}}{27}, N = \frac{\sqrt{6}}{54}$.

If we take $a = \frac{1}{2}, b = 2,\ \omega = \frac{1}{3}, d = 60\,000$, we get

$$f(t,h,k) < 115\,893\,505 \le \phi_3\left(\frac{d}{C}\right) = 54\times 10^8, \quad \text{for } 0\le t\le 1, 0\le h\le 75\,000, -60\,000\le k\le 60\,000;$$

$$f(t,h,k) > 17\,664 > \phi_3\left(\frac{b}{\omega N}\right) = 17\,496, \quad \text{for } \frac{1}{3}\le t\le \frac{2}{3}, 2\le h\le \frac{29}{3}, -60\,000\le k\le 60\,000;$$

$$f(t,h,k) < 0.281 < \phi_3\left(\frac{a}{L}\right) = 0.98, \quad \text{for } 0\le t\le 1, 0\le h\le \frac{1}{2}, -60\,000\le k\le 60\,000.$$

Then all conditions of Theorem 3.4 hold. Thus, with Theorem 3.4, problem (17) and (18) has at least three positive pseudo-symmetric solutions $u_1, u_2$ and $u_3$ which are symmetric on $[\frac{1}{3}, 1]$ with respect to $\frac{2}{3}$ such that

$$\max_{0\le t\le 1}|u_i'(t)| \le 60\,000 \quad \text{for } i=1,2,3,$$

$$2 < \min_{\frac{1}{3}\le t\le\frac{2}{3}}|u_1(t)|, \qquad \max_{0\le t\le 1}|u_1(t)| \le 75\,000,$$

$$\frac{1}{2} < \max_{0\le t\le 1}|u_2(t)| < \frac{29}{3}, \quad \text{with } \min_{\frac{1}{3}\le t\le\frac{2}{3}}|u_2(t)| < 2,$$

and

$$\max_{0\le t\le 1}|u_3(t)| < \frac{1}{2}.$$

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